Purposeful Academic Classes for Excelling Students Program
(Department of Education, Western Australia)

Session 2

Exponentials & Logarithms, Trigonometric Function
3.1.4 use trigonometric functions and their derivatives to solve practical problems

The second derivative and applications of differentiation
3.1.10 use the increments formula: $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x
3.1.11 apply the concept of the second derivative as the rate of change of the first derivative function
3.1.12 identify acceleration as the second derivative of position with respect to time
3.1.13 examine the concepts of concavity and points of inflection and their relationship with the second derivative
3.1.14 apply the second derivative test for determining local maxima and minima
3.1.15 sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection
3.1.16 solve optimisation problems from a wide variety of fields using first and second derivatives

Anti-differentiation
3.2.1 identify anti-differentiation as the reverse of differentiation
3.2.2 use the notation $\int f(x)dx$ for anti-derivatives or indefinite integrals
3.2.3 establish and use the formula $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$
3.2.4 establish and use the formula $\int e^x dx = e^x + c$
3.2.5 establish and use the formulas $\int \sin x\, dx = -\cos x + c$ and $\int \cos x\, dx = \sin x + c$
3.2.6 identify and use linearity of anti-differentiation
3.2.7 determine indefinite integrals of the form $\int f(ax - b)dx$
3.2.8 identify families of curves with the same derivative function
3.2.9 determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$

Calculus of the natural logarithmic function
4.1.12 establish and use the formula $\int \frac{1}{x}dx = \ln\ x\ + c$, for $x > 0$
4.1.13 determine derivatives of the form $\frac{d}{dx}(\ln f(x))$ and integrals of the form $\int \frac{f'(x)}{f(x)}dx$, for $f\ (x) > 0$
4.1.14 use logarithmic functions and their derivatives to solve practical problems

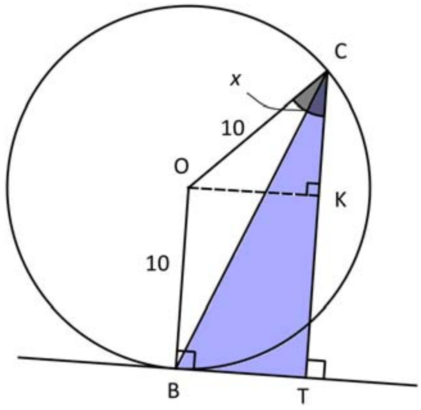
Definite integrals
3.2.10 examine the area problem and use sums of the form $\sum_i f(x_i)\ \delta x_i$ to estimate the area under the curve $y = f(x)$
3.2.11 identify the definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\sum_i f(x_i)\ \delta x_i$
3.2.12 interpret the definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$
3.2.13 interpret $\int_a^b f(x)dx$ as a sum of signed areas
3.2.14 apply the additivity and linearity of definite integrals

Fundamental theorem
3.2.15 examine the concept of the signed area function
$F(x) = \int_a^x f(t)dt$
3.2.16 apply the theorem: $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, and illustrate its proof geometrically
3.2.17 develop the formula $\int_a^b f'(x)dx = f(b) - f(a)$ and use it to calculate definite integrals

Applications of Differentiation

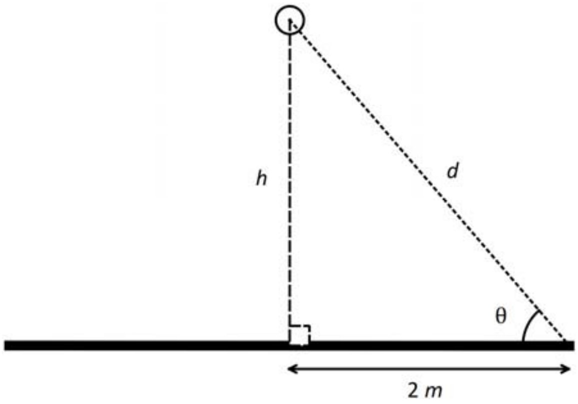
Worked Example 1 Calculator Assumed

The points B and C lie on a circle with centre at O and radius 10 cm. The line BT is a tangent to the circle at the point B. CT is perpendicular to BT. K is a point on CT such that OK is parallel to BT. $\angle OCT = x$ radians. Use a Calculus method to determine the exact value for x for which the area of ΔBCT is a maximum.



Worked Example 2 **Calculator Assumed**

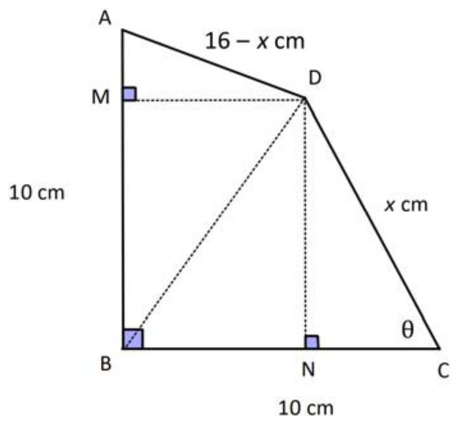
A lamp is hung *h m* above the centre of a circular table of radius 2 *m*.



The illuminance $E = k\,\dfrac{\sin\theta}{d^2}$ where *k* is a constant, *d* is the distance from the edge of the table to the lamp and θ is the angle with which light strikes the table at its edge. Use calculus to determine how high above the table the lamp should be hung to maximise the illuminance *E*.

Worked Example 3 **Calculator Assumed**

The accompanying diagram shows a quadrilateral ABCD with AB = BC = 10 cm. CD = x cm and AD = 16 – x cm. $\angle$ABC is a right angle. $\angle$BCD = θ radians. M and N are respectively the foot of the perpendiculars from D to AB and from D to BC.



(a) Given that $x = \dfrac{14}{8-5\sin\theta-5\cos\theta}$,

show that the area of quadrilateral ABCD is given by $A = 50 + \dfrac{70(\sin\theta-\cos\theta)}{8-5\sin\theta-5\cos\theta}$.

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(b) Verify that the area of is maximised when θ = 1.27209 radians.

Worked Example 4 **Calculator Assumed**

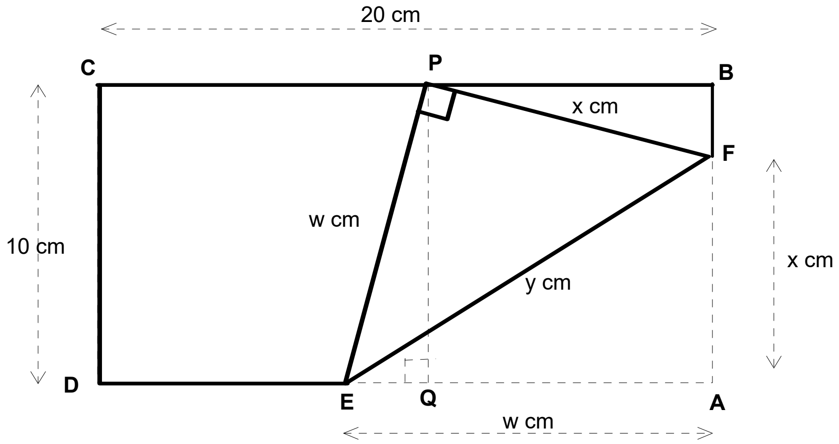
The cost *per hour* of running a transport vehicle is given by the function, $C = \$(\frac{v^2}{64} + 81)$, where v is the speed in km/h, and $0 < v < 100$.

(a) If the vehicle makes a 100 km journey with a constant speed of v, show that the *total cost*, of the journey is given by $T = \frac{25v}{16} + \frac{8100}{v}$.

(b) Use Calculus techniques to find the speed at which the *total cost* of the journey is minimized.

Worked Example 5 **Calculator Assumed**

The diagram below shows a rectangular piece of paper ABCD of dimensions 20 cm by 10 cm. The corner at A is folded along the crease EF so that it now touches point P as shown. Clearly EA = EP and FA = FP. EA = w cm, AF = x cm and EF = y cm.



(a) Show that $w = \dfrac{10x}{\sqrt{20x-100}}$

(b) Hence, find the value of x which will make the length of the crease a minimum. Give the exact minimum crease length. Show clearly the expressions you used and describe how you obtained your answer.

Anti-Differentiation

Commonly used Integrals

1. $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C \qquad n \neq -1$

2. $\int f'(x).[f(x)]^n \, dx = \dfrac{[f(x)]^{n+1}}{n+1} + C \qquad n \neq -1$

3. $\int \dfrac{f'(x)}{f(x)} \, dx = ln\left|f(x)\right| + C$

$\int \dfrac{1}{x} \, dx = ln\left|x\right| + C$ [special case]

4. $\int f'(x)e^{f(x)} \, dx = e^{f(x)} + C$

$\int e^{mx} \, dx = \dfrac{e^{mx}}{m} + C$ [special case]

5. $\int \cos(ax+b) \, dx = \dfrac{\sin(ax+b)}{a} + C$

6. $\int \sin(ax+b) \, dx = -\dfrac{\cos(ax+b)}{a} + C$

7. $\int \dfrac{1}{\cos^2(ax+b)} \, dx \equiv \int \sec^2(ax+b) \, dx = \dfrac{\tan(ax-b)}{a} + C$

8. $\int \dfrac{1}{\sin^2(ax+b)} \, dx \equiv \int \text{cosec}^2(ax+b) \, dx = \dfrac{-1}{\tan(ax+b)} \equiv -\cot(ax+b) + C$

Worked Example 6 **Calculator Free**

Determine each of the following;

(a) $\int (1-\frac{1}{2x^2})^2 \, dx$

(b) $\int \frac{x^2 - x^5}{2x^4} \, dx$

(c) $\int 4t(3+5t^2)^5 \, dt$

(d) $\int \frac{1}{4e^{5x+1}} \, dx$

(e) $\int \frac{3e^{-2x}}{(1+e^{-2x})^4} \, dx$

(f) $\int \frac{2x^3}{5e^{x^4}} \, dx$

Worked Example 7 **Calculator Free**

Find:

(a) $\int \sin(\pi x + \frac{\pi}{8})\, dx$

(b) $\int -\sin^2 x - \cos^2 x\, dx$

(c) $\int \cos x \sin^2 x \;\; dx$

(d) $\int \cos x \sqrt{1-\sin x} \;\; dx$

(e) $\int \frac{-3x^4 + 4x^2}{5x^5}\, dx$

(f) $\int \frac{-5x^2}{7+4x^3}\, dx$

Worked Example 8 **Calculator Free**

(a) Simplify $1-\dfrac{1}{x+1}$.

(b) Determine $\dfrac{d}{dx}\left(x\,ln(x+1)\right)$.

(c) Hence or otherwise evaluate $\int ln(x+1)\,dx$.

Worked Example 9 **Calculator Free**

$f(x)$ and $g(x)$ are continuous functions such that $f(x) > 0$ and $g(x) > 0$ for all values of x.

Determine with reasons if $\int \dfrac{f(x)}{g(x)}\,dx = \dfrac{\int f(x)\,dx}{\int g(x)\,dx}$.

Worked Example 10 **Calculator Free**

(a) Determine $\frac{d}{dx}x\cos(x)$.

(b) Determine $\frac{d}{dx}x^2\sin(x)$.

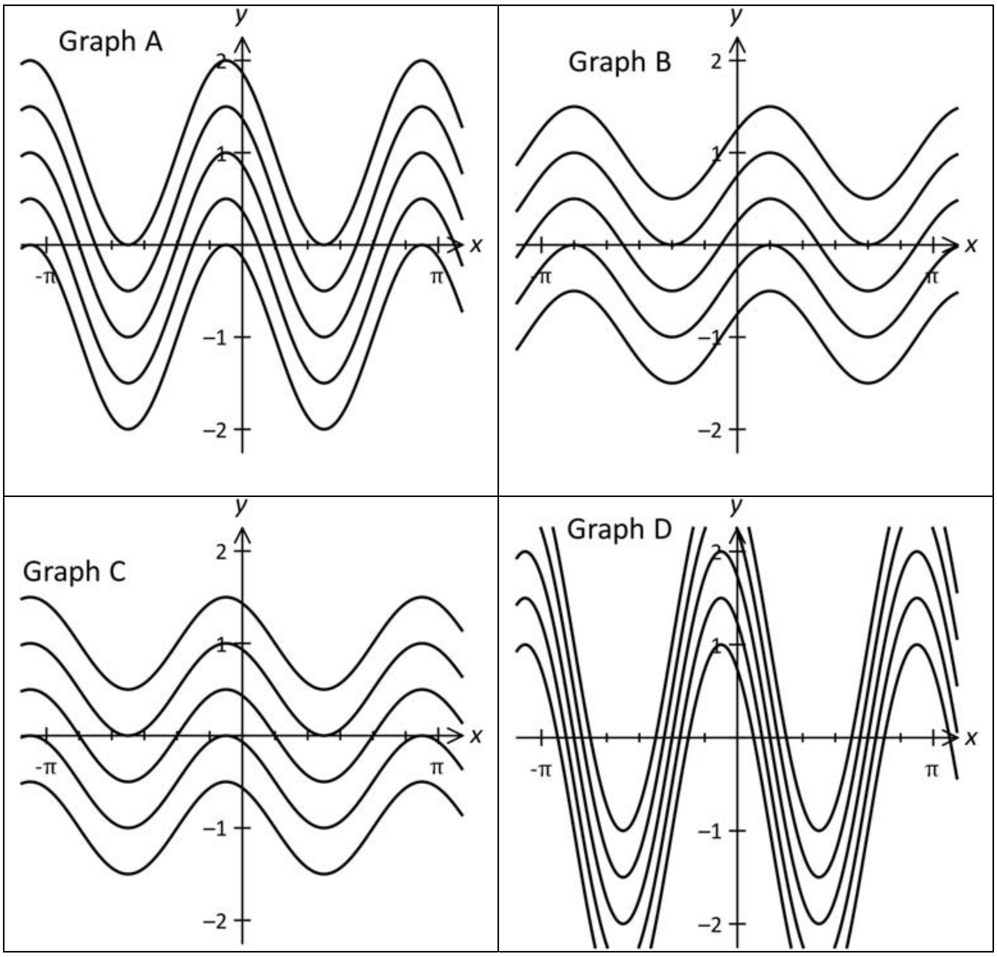
(c) Hence, or otherwise, determine $\int x^2\cos(x)\ dx$.

Worked Example 11 **Calculator Free**

(a) The curve $y = g(x)$ has gradient function $g'(x) = \cos\left(2x + \frac{\pi}{3}\right)$.

(i) Determine an expression for $y = g(x)$.

(ii) Graphs A, B, C and D display families of curves. Determine which graphs contain the family of curves $y = g(x)$ that correspond to $g'(x) = \cos\left(2x + \frac{\pi}{3}\right)$.



Graph ____________

Fundamental Theorem of Calculus

- Given that $F(x)$ is an anti-derivative of $f(x)$ and $f(x)$ is continuous in the interval $a \le x \le b$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
- If $f(t)$ is continuous in the interval $a \le t \le b$, then $\dfrac{d}{dx}\int_a^{g(x)} f(t)\,dt = f[g(x)].g'(x)$

Worked Example 12 **Calculator Free**

Find:

(a) $\int_0^1 -4t\,e^{t^2}(1+e^{t^2})\,dt$

(b) $\dfrac{d}{dx}\int_0^{e^{2x}} \sqrt{1+t^2}\,dt$

(c) $\dfrac{d}{dt}\int_{t^2}^{0} \dfrac{1-u}{1+u}\,du$

(d) $\int_0^4 \dfrac{d}{dt}\left[\dfrac{4-\sqrt{t}}{4+\sqrt{t}}\right]dt$

Worked Example 13 **Calculator Free**

Let $Q = \int_0^t \sin(\pi x^2)\, dx$ for $0 \le t \le 1.2$. Calculate the rate of change of Q at $t = \dfrac{\sqrt{2}}{2}$.

Worked Example 14 **Calculator Free**

Find the x-coordinate of the maximum point of the curve $y = \int_1^{x+2} (t-1)(t+2)e^t \, dt$.

Worked Example 15 **Calculator Free**

(a) Evaluate $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\tan 2x}\,dx$.

(b) Determine $\displaystyle\int_{0}^{2} \frac{1+2x+x^2}{1+x^2}\,dx$.

$\int_a^b f(x)\,dx$ as Sum of Signed Areas

Let $f(x)$ be continuous in the interval $a \le x \le b$.

- If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx$ represents the area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.
- In general, as $f(x)$ may criss-cross the x-axis several times, $\int_a^b f(x)\,dx$ represents the Sum of *signed areas* of the regions trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

Worked Example 16 **Calculator Assumed**

The function $y = f(x)$ is continuous for all real values of x.

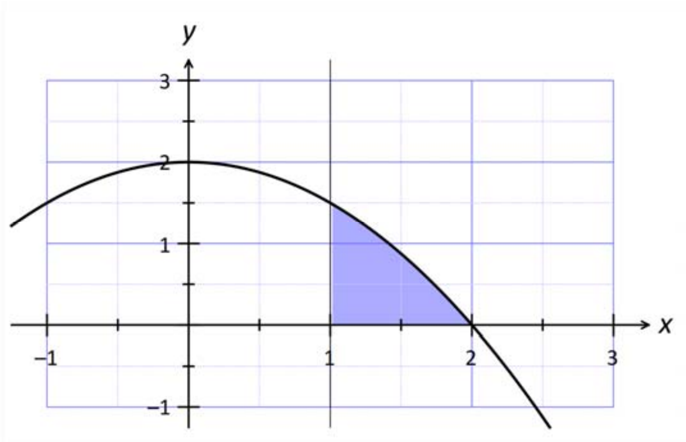
It is known that $\int_{-4}^{2} f(x)\,dx = A$ and $\int_{-4}^{12} f(x)\,dx = 0$ where A is a positive real number.

(a) Let R represent the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = -4$ and $x = 2$. Explain why the area of region $R \ge A$.

(b) Let S represent the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = 2$ and $x = 12$. If the area of region S is A, determine with reasons if $f(x) < 0$ for $2 \le x \le 12$.

Worked Example 17 **Calculator Assumed**

The shaded region in the diagram below is trapped between the curve $y = -0.5x^2 + 2$, the x-axis and the line $x = 1$.



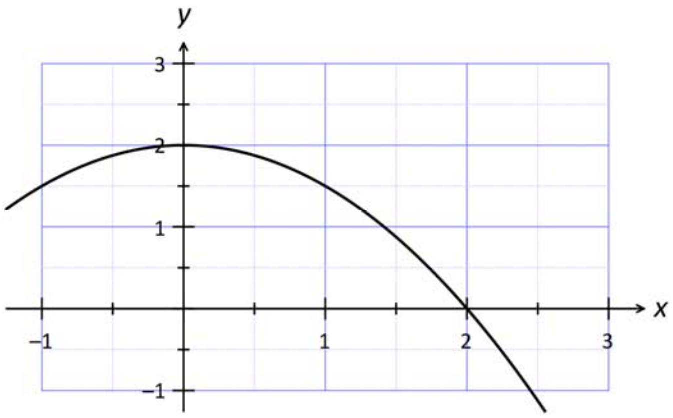
(a) The area of this region is to be estimated using 100 inscribed rectangular strips of uniform width. The height of the nth strip is $h = -0.5 \times (1 + 0.01n)^2 + 2$.

(i) State the area of the first strip.

(ii) Show that the area of this region using the 100 inscribed rectangular strips of uniform width is 0.825825.

(b) Determine the equation of the tangent to the curve at the point where $x = 1$.

(c) In the diagram below, draw the tangent to the curve at the point where $x = 1$.
Shade the region trapped between this tangent, the curve and the x-axis.



(d) Use the answer in (a) to estimate the area of the region shaded in (c).

Worked Example 18 **Calculator Free**

Given that $f(x)$ is continuous everywhere and $\int_{-4}^{6} f(x)\,dx = 20$ and $\int_{-4}^{10} f(x)\,dx = 5$, find:

(a) $\int_{6}^{-4} f(x)\,dx$

(b) $\int_{-3}^{11} 2f(x-1)\,dx$

(c) $\int_{4}^{-6} f(-x)+1\,dx$

(d) $\int_{6}^{10} 1-f(x)\,dx$

(e) $\int_{-2}^{3} f(2x)\,dx$

Worked Example 19 **Calculator Free**

Let $\frac{d}{dx}f(x) = g(x)$. The accompanying table provides the values for $f(x)$ and $g(x)$ for several values of x. Use the table given to answer the following questions.

x	-3	0	3
$f(x)$	25	1	-5
$g(x)$	-5	-8	7

(a) Calculate $\int_{-3}^{3} g(x)\,dx$.

(b) Calculate $\int_{0}^{3} g'(x)\,dx$.

(c) Calculate the value of $\frac{d}{dt}\int_{-3}^{t} g(x)\,dx$ when $t = -3$.

(d) Calculate the value of $\frac{d}{dx}e^{f(x)+5}$ when $x = 3$.